# INTEGRAL APPROXIMATION OF SIMPLICIAL VOLUME OF GRAPH MANIFOLDS

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ABSTRACT. Graph manifolds are manifolds that decompose along tori into pieces with a tame $S^1$-structure. In this paper, we prove that the simplicial volume of graph manifolds (which is known to be zero) can be approximated by integral simplicial volumes of their finite coverings. This gives a uniform proof of the vanishing of rank gradients, Betti number gradients and torsion homology gradients for graph manifolds.

## 1. INTRODUCTION

Many classical invariants from topology and group theory admit meaningful gradient invariants, which are defined by a stabilisation and normalisation process over finite coverings and finite index subgroups, respectively. For example, Betti number gradients coincide in many cases with the corresponding $L^2$-invariants [20].

We will consider an approximation problem for simplicial volume: The simplicial volume $\|M\|$ of an oriented closed connected (topological) $n$-manifold is the infimum of the $\ell^1$-norms of fundamental cycles with $\mathbb{R}$-coefficients of $M$. A related gradient invariant is the stable integral simplicial volume $\|M\|_{\mathbb{Z}}^{\infty}$ of $M$, defined as the infimum of the normalised integral simplicial volumes of finite coverings of $M$ (Section 3).

**Question 1.1** (integral approximation problem for simplicial volume)**.** *For which oriented closed connected manifolds M do we have*

$$\|M\| = \|M\|_{\mathbb{Z}}^{\infty} ?$$

In the present paper, we prove that the simplicial volume of graph manifolds satisfies integral approximation. We introduce a notion of graph manifolds as manifolds that decompose along tori into pieces that admit a tame $S^1$-structure (Section 2); our definition of graph manifolds excludes the case of spherical 3-manifolds, but it does include all other classical graph manifolds in dimension 3 as well as higher-dimensional examples.

**Theorem 1.2.** *Let M be an oriented closed connected graph manifold (in the sense of Definition 2.7) with residually finite fundamental group. Then*

$$\|M\| = \|M\|_{\mathbb{Z}}^{\infty} = 0.$$

*More generally: Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a descending chain of finite index subgroups of $\pi_1(M)$ with trivial intersection and let $(M_j)_{j\in\mathbb{N}}$ be the corresponding tower*

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*of finite coverings. Then*

$$\|M\| = \lim_{j\to\infty} \frac{\|M_j\|_{\mathbb{Z}}}{[\pi_1(M) : \Gamma_j]} = 0.$$

Vanishing of the ordinary simplicial volume follows already from work of Gromov [10], Yano [30], and Soma [25].

In addition, the following classes of manifolds are known to satisfy integral approximation for simplicial volume: closed surfaces of genus at least 1 [10], closed hyperbolic 3-manifolds [9], closed aspherical manifolds with residually finite amenable fundamental group [9], compact manifolds with "non-trivial" $S^1$-action [4], as well as certain glueings along tori [5]. In contrast, approximation fails uniformly for higher-dimensional hyperbolic manifolds [6] and it fails for closed manifolds with non-abelian free fundamental group [9, Remark 3.9].

**Question 1.3.** *Do all oriented closed connected* 3*-manifolds $M$ with infinite fundamental group satisfy the approximation identity $\|M\| = \|M\|_{\mathbb{Z}}^{\infty}$ for simplicial volume?*

**Applications.** Vanishing of stable integral simplicial volume, in particular, provides a uniform proof of the vanishing of the following gradient invariants:

**Corollary 1.4** (gradient invariants of graph manifolds)**.** *Let $M$ be an oriented closed connected graph manifold with residually finite fundamental group. Then the following hold:*

1. *The rank gradient of $\pi_1(M)$ is* 0*.*
2. *If $R$ is a principal ideal domain and $k \in \mathbb{N}$, then the* $\operatorname{rk}_R H_k(\,\cdot\,;R)$*-gradients of $M$ are* 0*.*
3. *If $k \in \mathbb{N}$, then the* $\log|\operatorname{tors} H_k(\,\cdot\,;\mathbb{Z})|$*-gradients of $M$ are* 0*.*
4. *The Euler characteristic of $M$ is* 0*.*

*Proof.* By Theorem 1.2, we have $\|M\|_{\mathbb{Z}}^{\infty} = 0$. The rank gradient estimate hence follows from the fact that stable integral simplicial volume is an upper bound for the rank gradient [18]. Alternatively, one can also apply the proof strategy for Theorem 1.2 via Theorem 1.6 to derive the triviality of the rank gradient from the corresponding results on cost [1, Theorem 1][15, Chapter 29–37].

For the homology and torsion homology gradients, we consider a descending chain $(\Gamma_j)_{j\in\mathbb{N}}$ of finite index subgroups of $\pi_1(M)$ with trivial intersection $\bigcap_{j\in\mathbb{N}} \Gamma_j$; because $\pi_1(M)$ is assumed to be residually finite, such chains do exist. Let $(M_j)_{j\in\mathbb{N}}$ be the corresponding tower of covering spaces of $M$. Then $\operatorname{rk}_R H_k(M_j;R) \leq \|M_j\|_{\mathbb{Z}}$ for all $j \in \mathbb{N}$ [9, Lemma 4.1] and hence (Theorem 1.2)

$$\limsup_{j\to\infty} \frac{\operatorname{rk}_R H_k(M_j;R)}{[\pi_1(M) : \Gamma_j]} \leq \lim_{j\to\infty} \frac{\|M_j\|_{\mathbb{Z}}}{[\pi_1(M) : \Gamma_j]} = 0.$$

Moreover, we have [9, proof of Theorem 1.6]

$$\limsup_{j\to\infty} \frac{\log|\operatorname{tors} H_k(M_j;R)|}{[\pi_1(M) : \Gamma_j]} \leq \log(n+1)\cdot\binom{n+1}{k+1}\cdot \lim_{j\to\infty} \frac{\|M_j\|_{\mathbb{Z}}}{[\pi_1(M) : \Gamma_j]}$$

(where $n := \dim M$) and the right-hand side is zero by Theorem 1.2.

The Euler characteristic is the alternating sum of the rational Betti numbers and it is multiplicative under finite coverings; therefore, the proof of vanishing of the $\dim_{\mathbb{Q}} H_*( \cdot ;\mathbb{Q})$-gradients shows that $\chi(M) = 0$. □

In dimension 3, these consequences can also be proved by a direct calculation [22, Theorem 3.14 and Corollary 3.18]. Furthermore, for 3-manifolds, Corollary 1.4 (3) is also a consequence of recent work by Lê [16].

*Remark* 1.5 ($L^2$-Betti numbers of graph manifolds). Let $M$ be an oriented closed connected graph manifold with residually finite fundamental group and $k \in \mathbb{N}$. Then the triviality of the $\dim_{\mathbb{Q}} H_k( \cdot ;\mathbb{Q})$-gradient of $M$ (Corollary 1.4) and Lück's approximation theorem [20] imply that

$$b_k^{(2)}(M) = 0.$$

Conversely, one can also prove the vanishing of the $L^2$-Betti numbers via $L^2$-methods (proceeding along the lines of our inductive proof of Theorem 1.6) and then deduce the vanishing of the $\dim_{\mathbb{Q}} H_k( \cdot ;\mathbb{Q})$-gradients via Lück's approximation theorem.

These results provide evidence for an affirmative answer to Gromov's question [11, p. 232] whether the vanishing of simplicial volume of aspherical closed manifolds implies the vanishing of their $L^2$-Betti numbers/Euler characteristic.

**Strategy of proof.** It is tempting to try to prove Theorem 1.2 by constructing good finite coverings by hand, taking advantage of the $S^1$-structure. However, the bookkeeping for the glueing steps would be quite tricky. It is much more efficient to pass to a more general setting with more flexible coefficient modules (see Section 3 for notation and terminology) that has better inheritance properties and allows for a straightforward induction proof. In this extended setting, we prove the following vanishing result:

**Theorem 1.6.** *Let $M$ be an oriented compact connected graph manifold with fundamental group $\Gamma$, and let $\alpha = \Gamma \curvearrowright X$ be an essentially free standard $\Gamma$-space. Then*

$$\left| M, \partial M \right|^{\alpha} = 0.$$

Theorem 1.6 can be proved by induction over the graph structure; the base case of a manifold with tame $S^1$-structure can be solved using methods by Fauser [4], the induction step requires a glueing argument similar to previous results by Fauser and Löh [5]. The main contribution of the present paper is to give a proper formalisation of this induction argument and adapting the glueing argument to the case of multiple boundary components; this will be done in the setting of fundamental groupoids and local coefficients.

Theorem 1.2 is then a consequence of the fact that taking the profinite completion of the fundamental group as coefficient action leads to the stable integral simplicial volume.

**Organisation of this article.** We first explain the setup of generalised graph manifolds (Section 2) and of the simplicial volumes needed for the proof of the main theorem (Section 3). The proof of Theorem 1.2 and Theorem 1.6 is given in Section 4.

## 2. Graph manifolds

Graph manifolds are (topological) manifolds that can be decomposed along tori into pieces that admit a tame $S^1$-structure. In order to keep track of such decompositions/glueings and facilitate induction proofs, we will use a more formal description via graphs.

### 2.1. Tame $S^1$-structures.

We will first introduce the basic building blocks of graph manifolds, namely manifolds that admit a tame $S^1$-structure.

Recall that a subspace $Y$ of a topological space $X$ is *$\pi_1$-injective* if for every basepoint $y \in Y$ the map $\pi_1(Y, y) \longrightarrow \pi_1(X, x)$ induced by the inclusion is injective.

**Definition 2.1** (tame $S^1$-structure)**.** A compact manifold $M$ of dimension $n \in \mathbb{N}$ admits a *tame $S^1$-structure* if there exists an $m \in \mathbb{N}$ and pairwise disjoint compact $n$-dimensional submanifolds $N_1, \dots, N_m$ of $M^\circ$ with the following properties:

- The complement $M' := M \setminus \bigcup_{j=1}^m N_j^\circ$ admits a smooth structure and a smooth $S^1$-fibre bundle structure $M' \longrightarrow B$ over a compact smooth $(n-1)$-manifold $B$ (possibly non-oriented and possibly with boundary) with $\pi_1$-injective (in $M$) fibres.
- For each $j \in \{1, \dots, m\}$ the manifold $N_j$ is homotopy equivalent to a torus of dimension at most $n-2$ and $N_j$ is a $\pi_1$-injective subspace of $M$.

*Example* 2.2. If $M$ is an aspherical oriented closed connected smooth manifold that has a smooth free $S^1$-action, then $M$ admits a tame $S^1$-structure in the sense of Definition 2.1: By the slice theorem, the canonical projection $M \longrightarrow M/S^1$ and the given $S^1$-action form an $S^1$-principal bundle over the smooth base manifold $M/S^1$ [27, Theorem 15.3.4]. Moreover, in this situation, the corresponding fibres (i.e., the orbits of the action) are $\pi_1$-injective [21, Corollary 1.43].

*Example* 2.3 (Seifert manifolds). Every compact Seifert 3-manifold that is *not* finitely covered by $S^3$ admits a tame $S^1$-structure. Indeed, for the $N_j$ we take the tubular neighbourhoods around the singular fibres provided by the definition of a Seifert 3-manifold. The regular fibres are $\pi_1$-injective [24, Lemma 3.2] and thus the first condition above is satisfied. Furthermore, by definition of a Seifert 3-manifold each $N_j$ contains a regular fibre $F$ such that the inclusion induced map $\mathbb{Z} \cong \pi_1(F) \longrightarrow \pi_1(N_j) \cong \mathbb{Z}$ is a monomorphism. It follows that the $N_j$ are also $\pi_1$-injective in $M$, thus the second condition above is satisfied.

### 2.2. Graphs.

We formalise the glueing of multiple manifolds with multiple boundary components via graphs. In the following, we will use unoriented graphs, possibly with multi-edges and loops. More precisely, we model graphs by their vertices, edges, and the map that assigns the incident vertices to the edges:

**Definition 2.4** (graph)**.** A *graph* is a triple $X = (V, E, \partial)$, where $V$ and $E$ are sets (the sets of *vertices* and *edges* of $X$, respectively) and $\partial$ is a map of the type

$$\partial\colon E \longrightarrow \bigl\{\{v, w\} \bigm| v, w \in V\bigr\}.$$

2.3. **Graphs of manifolds.** For the purpose of this paper, we will use the following terminology and conventions on graphs of manifolds. A graph of manifolds is a graph, where each vertex is decorated with a manifold with a tame $S^1$-structure and $\pi_1$-injective torus boundary components and where each edge describes a glueing of two of these boundary components. For $m \in \mathbb{N}$, we write $T^m := (S^1)^m$ for the standard $m$-torus. More precisely:

**Definition 2.5** (graph of manifolds)**.** Let $n \in \mathbb{N}$. A *graph of $n$-dimensional manifolds* is a triple $\Gamma = (X, (M_v)_{v \in V}, (f_e)_{e \in E})$ consisting of the following components:

- A graph $X = (V, E, \partial)$ in the sense of Definition 2.4 with finite sets $V$ and $E$.
- A family $(M_v)_{v \in V}$ of oriented connected $n$-manifolds that admit a tame $S^1$-structure (Definition 2.1) and whose boundary components are $\pi_1$-injective $(n-1)$-tori.
- A family $(f_e)_{e \in E}$ of maps with the following properties:
  - If $e \in E$ with $\partial e = \{v, w\}$ and $v \neq w$, then $f_e$ is a map (a homeomorphism onto its image)

    $$f_e\colon T^{n-1} \times \{v, w\} \longrightarrow \partial M_v \sqcup \partial M_w$$

    that induces an orientation-reversing homeomorphism between the corresponding boundary components of $M_v$ and $M_w$.
  - If $e \in E$ is a loop (i.e., $\partial e = \{v\}$ for some $v \in V$), then $f_e$ is a map

    $$f_e\colon T^{n-1} \times \{0, 1\} \longrightarrow \partial M_v$$

    that maps the two tori to different boundary components of $M_v$ and induces an orientation-reversing homeomorphism between these boundary components.
  - If $e, e' \in E$ satisfy $e \neq e'$, then the images of $f_e$ and $f_{e'}$ are disjoint.

Given a graph of manifolds, we can glue the vertex manifolds as specified by the edge maps:

**Definition 2.6** (geometric realisation)**.** Let $\Gamma = (X, (M_v)_{v \in V}, (f_e)_{e \in E})$ be a graph of $n$-manifolds with $X = (V, E, \partial)$. Then the *geometric realisation of* $\Gamma$ is the oriented $n$-manifold

$$M(\Gamma) := \Bigl(\coprod_{v \in V} M_v\Bigr) /{\sim},$$

where $\sim$ is the identification induced by the maps $(f_e)_{e \in E}$.

Alternatively, one can describe the geometric realisation as follows: Let $\Gamma = (X = (V, E, \partial), (M_v)_{v \in V}, (f_e)_{e \in E})$ be a graph of $n$-manifolds. Then the barycentric subdivision $X'$ of $X$ is a canonically oriented graph. The graph $\Gamma$ of manifolds gives rise to an $X'$-shaped diagram of manifolds by associating the torus with each of the new vertices obtained by barycentric subdivision and associating the (components of) the glueing maps with the edges. Then $M(\Gamma)$ is nothing but the colimit of this diagram.

Applying the functor $\pi_1$ (with a suitable treatment of basepoints) to a graph $\Gamma$ of manifolds gives rise to a graph of groups. Then the fundamental group of the geometric realisation $M(\Gamma)$ is isomorphic to the fundamental group of this graph of groups.

### 2.4. **Graph manifolds.**

**Definition 2.7** (graph manifold)**.** Let $n \in \mathbb{N}$. A *graph manifold* of dimension $n$ is a manifold homeomorphic to the geometric realisation of a graph of $n$-dimensional manifolds; recall that all vertex manifolds of such a graph admit a tame $S^1$-structure.

By construction, graph manifolds are orientable (through the orientation that is compatible with the vertex manifolds of an appropriate graph of manifolds), compact, and all boundary components are tori. Moreover, a graph manifold is connected if and only if the underlying graph is connected.

*Example* 2.8. This definition subsumes the following classes of graph manifolds:

- Graph manifolds in the sense of 3-manifolds, i.e., orientable prime 3-manifolds with empty or toroidal boundary such that all JSJ-components are Seifert manifolds [2], except for 3-manifolds that are finitely covered by $S^3$ (because Seifert JSJ-components are admissible vertex manifolds in our setting of graph manifolds, Example 2.3). Note that these manifolds all have residually finite fundamental group [13].
- Higher-dimensional graph manifolds in the sense of Frigerio, Lafont, Sisto [8] (because products of manifolds with $S^1$ are admissible vertex manifolds).

## 3. Simplicial volumes

Simplicial volumes of a manifold count the minimal number of singular simplices that are needed to represent the fundamental class, weighted by a norm on the coefficients.

### 3.1. **Simplicial volume.**

Classically, simplicial volume is defined with respect to constant coefficients [10, 17, 7]:

**Definition 3.1** (simplicial volume)**.** Let $n \in \mathbb{N}$ and let $M$ be an oriented compact connected $n$-manifold. Then the *simplicial volume* and the *integral* *simplicial volume* of $(M, \partial M)$ are defined by

$$\|M, \partial M\| := \inf\bigl\{|c|_1 \bigm| c \in C_n(M;\mathbb{R}) \text{ is a relative } \mathbb{R}\text{-fundamental cycle of } (M,\partial M)\bigr\},$$

$$\|M, \partial M\|_{\mathbb{Z}} := \inf\bigl\{|c|_1 \bigm| c \in C_n(M;\mathbb{Z}) \text{ is a relative } \mathbb{Z}\text{-fundamental cycle of } (M,\partial M)\bigr\}.$$

Here, $|\cdot|_1$ denotes the $\ell^1$-norm on $C_n(M;\mathbb{R})$ and $C_n(M;\mathbb{Z})$, respectively, associated with the basis of singular simplices.

The *stable integral simplicial volume* of $(M, \partial)$ is defined as

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} := \inf\Bigl\{\frac{\|N, \partial N\|_{\mathbb{Z}}}{d} \Bigm| d \in \mathbb{N}_{>0} \text{ and } N \text{ is a } d\text{-sheeted covering of } M\Bigr\}.$$

In the context of $L^2$-invariants or approximation questions for simplicial volume, we have to pass to twisted/local coefficients (Section 3.2).

3.2. **Normed local coefficients.** The definition of integral foliated simplicial volume involves twisted coefficients [12, 23, 19]. However, in the context of manifolds with multiple boundary components, it is more convenient to work in the framework of local coefficients than in the framework of twisted coefficients. Therefore, we will briefly recall local coefficients [26] and their relation with twisted coefficients.

**Definition 3.2** (normed local coefficient system)**.** Let $M$ be a topological space; we denote the fundamental groupoid of $M$ by $\pi(M)$ [27, Chapter 2.5]. A *normed local coefficient system on $M$* is a functor

$$\pi(M) \longrightarrow \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{sn}},$$

where $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{sn}}$ is the category of all semi-normed $\mathbb{Z}$-modules (and norm-non-increasing homomorphisms).

**Definition 3.3** (homology with local coefficients)**.** Let $M$ be a topological space and let $L\colon \pi(M) \longrightarrow \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{sn}}$ be a normed local coefficient system on $M$. The *chain complex $C_*(M; L)$ of $M$ with local coefficients in $L$* is defined as follows: For $n \in \mathbb{N}$, we set

$$C_n(M;L) := \bigoplus_{x\in M} \bigoplus_{\sigma \in S_n(M,x)} L(x)\cdot \sigma$$

where $S_n(M,x)$ is the set of all singular $n$-simplices $\sigma\colon \Delta^n \longrightarrow M$ in $M$ with $\sigma(e_0) = x$. Moreover, we define the boundary operator

$$\partial\colon C_n(M;L) \longrightarrow C_{n-1}(M;L)$$

by $\mathbb{Z}$-linear extension of

$$\partial(a\cdot\sigma) := \bigl(L(\sigma[0,1])\bigr)(a)\cdot\sigma[0] + \sum_{j=1}^{n}(-1)^j\cdot a\cdot\sigma[j]$$

for all $x \in M$, $a \in L(x)$, $\sigma \in S_n(M,x)$; here, $\sigma[j]$ denotes the $j$-th face of $\sigma$ and $\sigma[0,1]$ denotes the composition of $\sigma$ with the canonical parametrisation $[0,1] \longrightarrow \Delta^n$ of the $0-1$-edge of $\Delta^n$. Then $\partial\circ\partial = 0$.

We define the *$\ell^1$-semi-norm on $C_*(M;L)$* as follows: For all $n \in \mathbb{N}$ and every chain $\sum_{i=1}^k a_j \cdot \sigma_j \in C_n(M;L)$ in *reduced form*, i.e., the $\sigma_j$'s are pairwise distinct, let

$$\Big|\sum_{i=1}^k a_j \cdot \sigma_j\Big|_1 := \sum_{i=1}^k |a_j|_{L(\sigma_j(e_0))},$$

where $|\cdot|_{L(\sigma_j(e_0))}$ denotes the given semi-norm on the $\mathbb{Z}$-module $L\big(\sigma_j(e_0)\big)$.

The homology of $C_*(M;L)$ is called *homology of $M$ with local coefficients in $L$* and denoted by $H_*(M;L)$. The semi-norm on $C_*(M;L)$ induces the *$\ell^1$-semi-norm on $H_*(M;L)$* via

$$\|\alpha\|_1 := \inf\big\{|c|_1 \;\big|\; c \in C_n(M;L),\ \partial c = 0,\ [c] = \alpha \in H_n(M;L)\big\}$$

for all $n \in \mathbb{N}$ and all $\alpha \in H_n(M;L)$.

Let $U$ be a subspace of $M$. Let $I\colon \pi(U) \longrightarrow \pi(M)$ be the induced functor of the inclusion $U \subset M$ and let $L_U := L \circ I$. Then $C_*(U;L_U)$ is a subcomplex of $C_*(M;L)$ and we define the *chain complex of $M$ relative to $U$ with local coefficients in $L$* by

$$C_*(M,U;L) := C_*(M;L)/C_*(U;L_U);$$

the boundary operator $\partial$ on $C_*(M;L)$ induces a well-defined boundary operator on $C_*(M,U;L)$ that we denote by $\partial$ again. Then we define the *homology of $M$ relative to $U$ with local coefficients in $L$* by

$$H_*(M,U;L) := H_*\big(C_*(M,U;L)\big)$$

and we obtain the *$\ell^1$-semi-norm on $H_*(M,U;L)$* as follows: For all $n \in \mathbb{N}$ and all $\alpha \in H_n(M,U;L)$ let

$$||\alpha||_1 := \inf\big\{|c|_1 \;\big|\; c \in C_n(M;L), \partial c \in C_{n-1}(U;L_U), [c] = \alpha \in H_n(M,U;L)\big\}.$$

*Remark* 3.4 (local vs. twisted coefficients). Let $M$ be a path-connected topological space and let $x_0 \in M$; then $\operatorname{Aut}_{\pi(M)} x_0$ is nothing but the fundamental group $\pi_1(M,x_0)$ based at $x_0$.

- If $L\colon \pi(M) \longrightarrow \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{sn}}$ is a normed local coefficient system on $M$, then $L(x_0)$ has the structure of a normed right-$\pi_1(M,x_0)$-module.
- Conversely, if $A$ is a normed right-$\pi_1(M,x_0)$-module, then we can construct a local coefficient system $L_A$ on $M$ as follows: For all $x \in M \setminus \{x_0\}$, we choose a path $\gamma_x\colon [0,1] \longrightarrow M$ from $x_0$ to $x$ and we let $\gamma_{x_0}$ be the constant path at $x_0$.
  - For $x \in M$, we set $L_A(x) := A$.
  - For $[\gamma\colon x \to y]_* \in \pi(M)$, we set

$$\begin{aligned} L([\gamma]_*)\colon L_A(x) = A &\longrightarrow A = L_A(y) \\ a &\longmapsto a \cdot [\gamma_x * \gamma * \overline{\gamma}_y]_*. \end{aligned}$$

It is easy to verify that $L_{L(x_0)} \cong L$ and $L_A(x_0) = A$.

**Proposition 3.5** (homology with local vs. twisted coefficients)**.** *Let $M$ be a path-connected topological space that admits a universal covering $\pi_M\colon \widetilde{M} \longrightarrow M$, let $x_0 \in M$, and let $L\colon \pi(M) \longrightarrow \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{sn}}$ be a normed local coefficient system on $M$. Moreover, let $n \in \mathbb{N}$, let $D \subset \widetilde{M}$ be a set-theoretic fundamental domain for* *the deck transformation action of $\pi_1(M, x_0)$ on $\widetilde{M}$, and let $(\gamma_x)_{x\in M}$ be a family of paths connecting $x_0$ with the other points in $M$ (such that $\gamma_{x_0}$ is constant). Then*

$$\begin{aligned} C_n(M; L) &\longrightarrow L(x_0) \otimes_{\mathbb{Z}\pi_1(M,x_0)} C_n(\widetilde{M}; \mathbb{Z}) \\ a \cdot \sigma &\longmapsto L([\gamma_{\sigma(e_0)}]_*^{-1})(a) \otimes \widetilde{\sigma} \\ L(x_0) \otimes_{\mathbb{Z}\pi_1(M,x_0)} C_n(\widetilde{M}; \mathbb{Z}) &\longrightarrow C_n(M; L) \\ a \otimes \sigma &\longmapsto L([\gamma_{\pi_M \circ \sigma(e_0)}]_*)(a) \cdot \pi_M \circ \sigma \\ \text{with } \sigma(e_0) \in D & \end{aligned}$$

*are mutually inverse isomorphisms that do not increase the norm. Here, $\widetilde{\sigma}$ denotes the unique $\pi_M$-lift of $\sigma$ to $\widetilde{M}$ with $\widetilde{\sigma}(e_0) \in D$. These maps yield chain maps and hence induce mutually inverse isomorphisms*

$$H_k(M; L) \cong H_k\big(M; L(x_0)\big)$$

*that are isometric with respect to the induced semi-norms on homology.*

*Proof.* This is a straightforward calculation. □

3.3. **Relative integral foliated simplicial volume.** Integral foliated simplicial volume is a simplicial volume that uses dynamical systems of the fundamental groupoid as local coefficients. For basic notions on (actions) on standard Borel probability spaces we refer to the literature [14, 15].

**Definition 3.6** (parameter spaces)**.** Let $G$ be a groupoid. A *standard $G$-space* is functor $G \longrightarrow \mathsf{SBP}$ into the category of standard Borel probability spaces (with probability measure preserving transformations). A standard $G$-space is *essentially free [ergodic]*, if for every point in $G$ the induced group action is essentially free [ergodic]. Recall that a group action is *essentially free* almost all points have trivial isotropy.

**Definition 3.7** (normed local coefficients associated with standard actions)**.** Let $G$ be a groupoid and let $\alpha\colon G \longrightarrow \mathsf{SBP}$ be a standard $G$-space. We define the associated normed local coefficient system $L^\infty(\alpha; \mathbb{Z})\colon G \longrightarrow \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{sn}}$ by

$$L(x) := L^\infty(\alpha(x); \mathbb{Z})$$

(with the $L^1$-"norm") for all points $x$ of $G$ and

$$\begin{aligned} L(g)\colon L^\infty(\alpha(x); \mathbb{Z}) &\longrightarrow L^\infty(\alpha(y); \mathbb{Z}) \\ f &\longmapsto f \circ \alpha(g^{-1}) \end{aligned}$$

for all morphisms $g\colon x \longrightarrow y$ of $G$.

**Definition 3.8** (parametrised relative fundamental class)**.** Let $n \in \mathbb{N}$, let $M$ be an oriented compact $n$-manifold (possibly with boundary), and let $\alpha$ be a standard $\pi(M)$-space. Then the image

$$[M, \partial M]^\alpha \in H_n\big(M, \partial M; L^\infty(\alpha; \mathbb{Z})\big)$$

of the integral fundamental class $[M, \partial M]_{\mathbb{Z}} \in H_n(M, \partial M; \mathbb{Z})$ under the change of coefficients map induced by the inclusion of (the constant system) $\mathbb{Z}$ into $L^\infty(\alpha; \mathbb{Z})$ (as constant functions) is the *$\alpha$-parametrised (relative) fundamental class of $(M, \partial M)$*.

**Proposition 3.9.** *Let $n \in \mathbb{N}$, let $M$ be an oriented compact $n$-manifold (possibly with boundary), and let $\alpha$ be a standard $\pi(M)$-space. Then*

$$\partial\bigl([M,\partial M]^\alpha\bigr) = [\partial M]^{\operatorname{res}^{\pi(M)}_{\pi(\partial M)} \alpha} \in H_{n-1}(\partial M; \operatorname{res}^{\pi(M)}_{\pi(\partial M)} \alpha).$$

*Here, $\operatorname{res}^{\pi(M)}_{\pi(\partial M)} \alpha$ is the composition of $\alpha \colon \pi(M) \longrightarrow \mathsf{SBP}$ with the groupoid morphism $\pi(\partial M) \longrightarrow \pi(M)$ induced by the inclusion $\partial M \longrightarrow M$.*

*Proof.* We only need to check this equality for integral (relative) fundamental classes, where it is well-known. □

**Definition 3.10** (relative integral foliated simplicial volume)**.** Let $n \in \mathbb{N}$, let $M$ be an oriented compact $n$-manifold (possibly with boundary), and let $\alpha$ be a standard $\pi(M)$-space. Then the *$\alpha$-parametrised simplicial volume of* $(M,\partial M)$ is defined by

$$\bigl| M,\partial M \bigr|^\alpha := \inf\bigl\{ |c|_1 \bigm| c \in C_n(M;\alpha) \text{ represents } [M,\partial M]^\alpha \bigr\};$$

recall that if $\alpha = \pi(M) \curvearrowright (X,\mu)$ and $c = \sum_{j=1}^k f_j \cdot \sigma_j \in C_n(M;\alpha)$ is in reduced form, then

$$|c|_1 = \sum_{j=1}^k \int_X |f_j| \, d\mu.$$

The *relative integral foliated simplicial volume* $\bigl| M,\partial M \bigr|$ *of* $(M,\partial M)$ is the infimum over all parametrised simplicial volumes of $(M,\partial M)$ (the isomorphism types of standard $\pi(M)$-spaces form a set).

This definition is compatible with the usual definition of parametrised and integral foliated simplicial volume in terms of twisted coefficients [23, 19]. For simplicity, we only formulate this in the closed case:

**Proposition 3.11** (comparison with the twisted definition)**.** *Let $n \in \mathbb{N}$, let $M$ be an oriented closed connected $n$-manifold, let $x_0 \in M$, and let $\alpha$ be a standard $\pi(M)$-space. Then $\bigl| M \bigr|^\alpha$ coincides with the $\ell^1$-semi-norm of the parametrised fundamental class in homology $H_n(M; L^\infty(\alpha(x_0);\mathbb{Z}))$ with twisted coefficients in the $\mathbb{Z}\pi_1(M,x_0)$-module $L^\infty(\alpha(x_0);\mathbb{Z})$.*

*Proof.* This is a special case of Proposition 3.5. □

Furthermore, the previous proposition also extends to the case of manifolds with boundary. So, in principle, one could always get away with the twisted version. However, working with twisted coefficients requires the choice of a basepoint. When dealing with manifolds with multiple boundary components, this leads to an unpleasant overhead.

3.4. **A local criterion.** Given a top-dimensional parametrised homology class, we will need a local criterion that decides whether this class coincides with the parametrised fundamental class or not. As a first step, we briefly recall parametrised Poincaré-Lefschetz duality (which, in particular, allows to compute the top-dimensional parametrised homology).

$$
\begin{array}{ccccc}
H_n(U,\partial U;\mathbb{Z}) & \longrightarrow & H_n(U,\partial U;L^\infty(\alpha';\mathbb{Z})) & \longrightarrow & H_n(U,U\setminus V^\circ;L^\infty(\alpha';\mathbb{Z})_{\pi(U)}) \\
\cong\downarrow & & \downarrow\varphi & & \downarrow \\
H_n(M,M\setminus U^\circ;\mathbb{Z}) & \longrightarrow & H_n(M,M\setminus U^\circ;L^\infty(\alpha;\mathbb{Z})) & \longrightarrow & H_n(M,M\setminus V^\circ;L^\infty(\alpha;\mathbb{Z})_{\pi(M)}) \\
\cong\uparrow & & \uparrow & & \| \\
H_n(M,\partial M;\mathbb{Z}) & \longrightarrow & H_n(M,\partial M;L^\infty(\alpha;\mathbb{Z})) & \longrightarrow & H_n(M,M\setminus V^\circ;L^\infty(\alpha;\mathbb{Z})_{\pi(M)})
\end{array}
$$

FIGURE 1. Proving the local criterion for parametrised fundamental classes

**Proposition 3.12** (parametrised Poincaré-Lefschetz duality). *Let $n \in \mathbb{N}$, let $M$ be an oriented compact connected $n$-manifold (possibly with boundary), and let $\alpha$ be a standard $\pi(M)$-space. Then the relative cap-product induces isomorphisms*

$$\cdot \cap [M]_{\mathbb{Z}} \colon H^{n-k}\big(M;L^\infty(\alpha;\mathbb{Z})\big) \longrightarrow H_k\big(M,\partial M;L^\infty(\alpha;\mathbb{Z})\big)$$

*for all $k \in \mathbb{N}$.*

*Proof.* This is a special case of Poincaré-Lefschetz duality with twisted coefficients: If $M$ is triangulable, the pair $(M,\partial M)$ is a connected Poincaré pair in the sense of Wall [28, Theorem 2.1] and connected Poincaré pairs satisfy Poincaré-Lefschetz duality with twisted coefficients [29, Lemma 1.2].

For the general case of topological manifolds one can, for example, adapt the classical proof [3, Chapter VI] to the setting of twisted coefficients. □

**Proposition 3.13** (local criterion). *Let $n \in \mathbb{N}$, let $M$ be an oriented compact $n$-manifold (possibly with boundary), let $\alpha$ be a standard $\pi(M)$-space, let $U \subset M^\circ$ be a non-empty compact connected $n$-dimensional submanifold (possibly with boundary), and let $c \in C_n(M;L^\infty(\alpha;\mathbb{Z}))$ be a relative cycle of $(M,\partial M)$. Then the following are equivalent:*

1. *The chain $c$ is an $\alpha$-parametrised relative fundamental cycle of $(M,\partial M)$.*
2. *In $H_n\big(M,M\setminus U^\circ;L^\infty(\alpha;\mathbb{Z})\big)$, the chain $c$ represents the class $\varphi[U,\partial U]_{\alpha'}$, where $\alpha'$ is the restriction of $\alpha$ to $\pi(U)$ and*

$$\varphi\colon H_n\big(U,\partial U;L^\infty(\alpha';\mathbb{Z})\big) \longrightarrow H_n\big(M,M\setminus U^\circ;L^\infty(\alpha;\mathbb{Z})\big)$$

*is induced by the canonical transformation $L^\infty(\alpha';\mathbb{Z}) \longrightarrow L^\infty(\alpha;\mathbb{Z})$.*

*Proof.* The inclusions $(M,\partial M) \longrightarrow (M,M\setminus U^\circ)$, $(U,\partial U) \longrightarrow (M,M\setminus U^\circ)$ give rise to the left hand side of the commutative diagram in Figure 1.

For the right hand side, we proceed as follows: Let $V \subset U^\circ \subset M$ be an embedded closed $n$-ball. The local coefficient systems $L^\infty(\alpha;\mathbb{Z})_{\pi(M)}$ etc. are the coinvariants of the original systems $L^\infty(\alpha;\mathbb{Z})$, i.e., they are obtained by dividing out the action of the automorphisms (that is the fundamental groups) at each point. Hence, by construction, these local coefficient systems can be viewed as constant coefficients and thus lead to ordinary homology groups.

If $c$ satisfies the first condition, then $c$ represents $\varphi[U,\partial U]_{\alpha'}$ in the relative group $H_n(M,M\setminus U^\circ;L^\infty(\alpha;\mathbb{Z}))$ because the isomorphisms of the leftmost column in Figure 1 are compatible with the corresponding integral fundamental classes.

Conversely, let $c$ represent $\varphi[U, \partial U]_{\alpha'}$ in $H_n(M, M \setminus U^\circ; L^\infty(\alpha; \mathbb{Z}))$. Then $c$ represents the image of the ordinary fundamental class $[U, U \setminus V^\circ]_\mathbb{Z}$, and thus of $[M, M \setminus V^\circ]_\mathbb{Z}$, in $H_n(M, M \setminus V^\circ; L^\infty(\alpha; \mathbb{Z})_{\pi(M)})$. Therefore, $c$ satisfies the hypothesis of the local criterion for embedded balls [4, Proposition 3.9]; it should be noted that the version of the cited local criterion can also for local coefficients be derived from parametrised Poincaré-Lefschetz duality (Proposition 3.12) in the same way as for twisted coefficients. Applying this local criterion to the ball $V \subset M$ then implies that $c$ is an $\alpha$-parametrised fundamental cycle of $(M, \partial M)$. □

## 4. Simplicial volumes of graph manifolds

We will first prove Theorem 1.6 and then we will derive Theorem 1.2. In order to prove Theorem 1.6, one can either perform all glueings at once or only glue along one torus at a time (and then proceed by induction). We prefer the latter version. When combining chains along tori, we need a way to fill boundaries efficiently:

**Proposition 4.1** (parametrised UBC for tori). *Let $n \in \mathbb{N}_{>0}$, let $G := \pi(T^n)$, and let $\alpha$ be an essentially free standard $G$-space. Then $C_*(T^n; \alpha)$ satisfies the uniform bounday condition (UBC) in every degree, i.e.: For every $k \in \mathbb{N}$ there is a constant $K \in \mathbb{R}_{>0}$ such that for every null-homologous cycle $c \in C_k(T^n; \alpha)$ there exists a chain $b \in C_{k+1}(T^n; \alpha)$ with*

$$\partial b = c \quad \text{and} \quad |b|_1 \leq K \cdot |c|_1.$$

*Proof.* By the correspondence between local and twisted coefficients on the chain level (Proposition 3.5), this is a direct consequence of the parametrised uniform boundary condition for tori formulated in terms of twisted coefficients [5, Theorem 1.3]. □

### 4.1. Vertex manifolds.

We first treat the base case of $S^1$-bundles; in a second step, we then use a glueing argument and UBC to treat the case of general tame $S^1$-structures.

**Proposition 4.2.** *Let $M$ be an oriented compact connected smooth $n$-manifold that is the total space of a smooth $S^1$-bundle $p\colon M \longrightarrow B$ over a compact smooth $(n-1)$-manifold $(B, \partial B)$. Then*

$$\left| M, \partial M \right|^{\alpha} = 0$$

*holds for all standard $\pi(M)$-spaces $\alpha$ whose restrictions to all fibres are essentially free.*

*Proof.* We choose a triangulation of $B$ that is fine enough such that the bundle $p$ is trivial over every simplex in this triangulation. As in Yano's proof for vanishing of simplicial volume of oriented closed connected smooth manifolds with non-trivial smooth $S^1$-action [30], we define a sequence

$$M_{n-1} \xrightarrow{p_{n-2}} M_{n-2} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{p_0} M_0 := M$$

of hollowings of $M$. Let $X_0 := p^{-1}(B^{(0)})$ be the pre-image of the 0-skeleton of $B$. We define $p_0$ to be the hollowing at $X_0 \subset M_0$ [30, Section 2], i.e., we obtain $M_1$ from $M_0$ by removing a (small) tubular neighbourhood of $X_0$. Now, we inductively define for all $j \in \{1, \dots, n-1\}$ the map $p_j$ to be the hollowing at $X_j \subset M_j$, where $X_j$ is the pullback of the $j$-skeleton of $B$ along $p \circ p_0 \circ p_1 \circ \dots \circ p_{j-1}$. Let $B^{[n-1]}$ denote the set of $(n-1)$-simplices in the chosen triangulation of $B$. Furthermore, for every $\Delta \in B^{[n-1]}$ let

$$\Delta_{n-1} \longrightarrow \Delta_{n-2} \longrightarrow \cdots \longrightarrow \Delta_1 \longrightarrow \Delta_0 = \Delta$$

be the induced sequence of restricted hollowings at the skeleta of $\Delta$. Then

$$M_{n-1} \cong \coprod_{\Delta \in B^{[n-1]}} \Delta_{n-1} \times S^1$$

and $M_{n-1}$ inherits an orientation from $M$.

Let $\alpha$ be a standard $\pi(M)$-space whose restrictions to the fibres are essentially free. We set $\alpha_0 := \alpha$. For every $j \in \{1, \dots, n-1\}$ let $\alpha_j$ be the standard $\pi(M_j)$-space that is given by restricting $\alpha$ along $p_0 \circ p_1 \circ \dots \circ p_{j-1}$. Furthermore, we denote by $P_j$ the induced map of $p_j$ from the $\alpha_j$-parametrised chain complex of $M_j$ to the $\alpha_{j-1}$-parametrised chain complex of $M_{j-1}$.

For every simplex $\Delta \in B^{[n-1]}$, we choose an integral relative fundamental cycle $z_\Delta \in C_{n-1}(\Delta_{n-1}; \mathbb{Z})$ that is compatible with the CW-structure on $\partial\Delta$ given by the sequence of hollowings above. By hypothesis, the restriction of $\alpha_{n-1}$ to each $\Delta \times S^1$ yields an essentially free standard $\pi(S^1)$-space $\alpha_\Delta$. Let $\varepsilon \in \mathbb{R}_{>0}$. Then, for every $\Delta \in B^{[n-1]}$ there exists an $\alpha_\Delta$-parametrised fundamental cycle $c_\Delta^{S^1}$ of $S^1$ with

$$|c_\Delta^{S^1}|_1 < \varepsilon$$

such that $z_\Delta \times c_\Delta^{S^1}$ is a $\alpha_\Delta$-parametrised relative fundamental cycle of $\Delta \times S^1$ (with the orientation inherited from $M$); Schmidt [23, Proposition 5.30] stated this for ergodic parameter spaces, but his proof also works for essentially free parameter spaces. We set

$$z := \sum_{\Delta \in B^{[n-1]}} z_\Delta \times c_\Delta^{S^1} \in C_n(M_{n-1}; \alpha_{n-1}).$$

Let $A := \max\{|z_\Delta|_1 \mid \Delta \in B^{[n-1]}\}$. Then we have

$$|z|_1 \le n \cdot |B^{[n-1]}| \cdot A \cdot \varepsilon.$$

Therefore, $z$ is a parametrised relative fundamental cycle of $M_{n-1}$ with small norm. Starting with $z$, we wish to construct a parametrised relative fundamental cycle of $M$ of small norm.

Indeed, we are now in the same situation as in the proof of the analogous vanishing result for parametrised simplicial volumes of smooth manifolds with non-trivial smooth $S^1$-actions [4, Theorem 1.1, Remark 6.4], the only difference being that we had to do one more step in the sequence of hollowings to obtain a trivial $S^1$-bundle. We then proceed as in the cited proof, adapting the chain

$$P_0 \circ P_1 \circ \dots \circ P_{n-2}(z) \in C_n(M; \alpha)$$

to get an $\alpha$-parametrised relative fundamental cycle of $M$ without increasing the norm too much: We inductively investigate the defect of the push-forward of $z$ in $M_j$ from being a parametrised relative fundamental cycle and we fill the defect with the help of Proposition 4.1. □

**Proposition 4.3** (vertex manifolds)**.** *Let $M$ be an oriented connected compact manifold that admits a tame $S^1$-structure. Then*

$$\left| M, \partial M \right|^{\alpha} = 0$$

*holds for all essentially free standard $\pi(M)$-spaces $\alpha$.*

*Proof.* Let $n := \dim M$. Because $M$ admits a tame $S^1$-structure, there exists an $m \in \mathbb{N}$ and pairwise disjoint compact submanifolds $N_1, \dots, N_m$ (with boundary) of dimension $n$ with the following properties (Definition 2.1):

- The complement $M' := M \setminus \bigcup_{j=1}^{m} N_j^\circ$ admits a smooth $S^1$-bundle structure.
- For each $j \in \{1, \dots, m\}$ the manifold $N_j$ is homotopy equivalent to a torus of dimension at most $n - 2$ and $N_j$ is a $\pi_1$-injective subspace of $M$.

Let $\alpha$ be an essentially free standard $\pi(M)$-space and let $\alpha'$ be the induced standard $\pi(M')$-space. Because the fibres are $\pi_1$-injective, the restriction of $\alpha'$ to each fibre is essentially free. Therefore, $M'$ and $\alpha'$ satisfy the hypotheses of Proposition 4.2.

Let $\varepsilon \in \mathbb{R}_{>0}$. By Proposition 4.2, there exists a chain $z' \in C_n(M'; \alpha')$ representing $[M', \partial M']_{\alpha'}$ with

$$|z'|_1 \leq \varepsilon.$$

For $j \in \{1, \dots, m\}$ let $z_j := (\partial z')|_{N_j} \in C_{n-1}(N_j; \alpha_j)$ be the $N_j$-component of $\partial z'$. Here, $\alpha_j$ denotes the restriction of $\alpha$ to $N_j \subset M' \subset M$; because $\alpha$ is essentially free and $N_j$ is a $\pi_1$-injective subspace of $M$, also $\alpha_j$ is an essentially free standard $\pi(N_j)$-space. As $N_j$ is homotopy equivalent to a torus of dimension at most $n - 2$, the chain $z_j$ is null-homologous in $C_*(N_j; \alpha_j)$ for dimension reasons. Hence, we can apply Proposition 4.1 (and homotopy invariance of UBC [4, Proposition 3.15]) to obtain a chain $b_j \in C_n(N_j; \alpha_j)$ with

$$\partial b_j = z_j \quad \text{and} \quad |b_j|_1 \leq K_j \cdot |z_j|_1$$

(where $K_j$ is a UBC-constant for $C_{n-1}(N_j; \alpha_j)$). We now set

$$z := z' - \sum_{j=1}^{m} b_j \in C_n(M; \alpha)$$

(using the obvious inclusions between the parametrised chain complexes). By construction, we have

$$|z|_1 \leq \varepsilon + \sum_{j=1}^{m} K_j \cdot (n+1) \cdot \varepsilon \quad \text{and} \quad \partial z = \partial z'.$$

The local criterion (Proposition 3.13) shows that $z$ represents $[M, \partial M]_\alpha$ (because $z$ restricts to the relative fundamental cycle $z'$ of $(M', \partial M')$). Therefore,

$$\left| M, \partial M \right|^{\alpha} \leq |z|_1 \leq \varepsilon + \sum_{j=1}^{m} K_j \cdot (n+1) \cdot \varepsilon.$$

Taking $\varepsilon \to 0$ proves $\left| M, \partial M \right|^{\alpha} = 0$. □

For non-spherical Seifert 3-manifolds it was already known that the stable integral simplicial volume is 0 [19]. However, for the induction step we will need a more general vanishing result than just for stable integral simplicial volume. Therefore, already for the treatment of graph manifolds in dimension 3 the dynamical version of simplicial volume is helpful.

4.2. **Edge glueings.**

**Proposition 4.4** (glueings along tori)**.** *Let $n \in \mathbb{N}_{\geq 2}$ and let $(M_1, \partial M_1)$ and $(M_2, \partial M_2)$ be oriented compact connected n-manifolds with boundary. Let $T_1 \subset \partial M_1$ and $T_2 \subset \partial M_2$ be $\pi_1$-injective components of $\partial M_1$ and $\partial M_2$, respectively, that are homeomorphic to the torus $T^{n-1}$. Let $f \colon T_1 \longrightarrow T_2$ be an orientation-reversing homeomorphism, let*

$$M := M_1 \cup_f M_2$$

*be the oriented compact connected n-manifold obtained by glueing $M_1$ and $M_2$ along the boundary components $T_1$ and $T_2$ via $f$, let $G := \pi(M_1) *_{\pi(f)} \pi(M_2)$ be the corresponding pushout groupoid, and let $\alpha$ be an essentially free standard G-space with*

$$\left| M_1, \partial M_1 \right|^{\operatorname{res}^G_{\pi(M_1)} \alpha} = 0 \quad \text{and} \quad \left| M_2, \partial M_2 \right|^{\operatorname{res}^G_{\pi(M_2)} \alpha} = 0.$$

*Then $\left| M, \partial M \right|^{\alpha} = 0$. In particular, $\left| M, \partial M \right| = 0$.*

*Proof.* We proceed as in the case with a single boundary component [5, Proposition 10.3]: In order to keep the notation simple, we view $M_1$ and $M_2$ as subspaces of $M$ and identify $T_1$ and $T_2$ via $f$.

By the Seifert and van Kampen theorem for fundamental groupoids, the inclusions of $M_1$ and $M_2$ into $M$ induce an isomorphism $G \cong \pi(M)$. Moreover, as the boundary components are $\pi_1$-injective, we also know that the canonical maps $\pi(M_1) \longrightarrow \pi(M)$ and $\pi(M_2) \longrightarrow \pi(M)$ are injective at every base-point. Therefore, the restrictions $\alpha_1 := \operatorname{res}^G_{\pi(M_1)} \alpha$ and $\alpha_2 := \operatorname{res}^G_{\pi(M_2)} \alpha$ of the essentially free $G$-space $\alpha$ are essentially free; hence, also $\alpha_0 := \operatorname{res}^G_{\pi(T_1)} \alpha = \operatorname{res}^G_{\pi(T_2)} \alpha$ is essentially free.

Let $K \in \mathbb{R}_{>0}$ be an $(n-1)$-UBC constant for $C_*(T_1; \alpha_0)$ (Proposition 4.1). Let $\varepsilon \in \mathbb{R}_{>0}$. Because of $\left| M_1, \partial M_1 \right|^{\alpha_1} = 0$ and $\left| M_2, \partial M_2 \right|^{\alpha_2} = 0$ there exist parametrised relative fundamental cycles $c_1 \in C_n(M_1; \alpha_1)$ as well as $c_2 \in C_n(M_2; \alpha_2)$ with

$$|c_1|_1 \leq \varepsilon \quad \text{and} \quad |c_2|_1 \leq \varepsilon.$$

Then

$$c_0 := (\partial c_1)|_{T_1} + (\partial c_2)|_{T_2} \in C_{n-1}(T_1; \alpha_0)$$

is a null-homologous cycle in $C_{n-1}(T_1;\alpha_0)$ (because the glueing map $f$ is orientation-reversing and $(\partial c_1)|_{T_1}$ and $(\partial c_2)|_{T_2}$ are $\alpha_0$-parametrised fundamental cycles of $T_1$ by Proposition 3.9); by construction,

$$|c_0|_1 \leq 2 \cdot (n+1) \cdot \varepsilon.$$

By the uniform boundary condition on the torus $T_1 = T_2$, there exists a chain $b \in C_n(T_1;\alpha_0)$ with

$$\partial b = c_0 \quad \text{and} \quad |b|_1 \leq K \cdot |c_0|_1.$$

Then $c := c_1 + c_2 - b \in C_n(M;\alpha)$ is a cycle that satisfies

$$|c|_1 \leq 2 \cdot \varepsilon + K \cdot 2 \cdot (n+1) \cdot \varepsilon.$$

Moreover, $c$ is an $\alpha$-parametrised relative fundamental cycle of $(M, \partial M)$ (Proposition 3.13).

Taking the infimum over all $\varepsilon \in \mathbb{R}_{>0}$ shows that $\left| M, \partial M \right|^{\alpha} = 0$. □

**Proposition 4.5** (self-glueing along tori). *Let $n \in \mathbb{N}_{\geq 2}$ and let $(M, \partial M)$ be an oriented compact connected $n$-manifold with boundary. Let $T_1, T_2 \subset \partial M$ be two different $\pi_1$-injective components of $\partial M$ that are homeomorphic to the torus $T^{n-1}$. Let $f \colon T_1 \longrightarrow T_2$ be an orientation-reversing homeomorphism, let*

$$N := M/(T_1 \sim_f T_2)$$

*be the oriented compact connected $n$-manifold obtained by glueing $M$ to itself along $T_1, T_2$ via $f$, let $G := \pi(M)*_{\pi(f)}$ be the corresponding HNN-extension groupoid, and let $\alpha$ be an essentially free standard $G$-space with*

$$\left| M, \partial M \right|^{\operatorname{res}^G_{\pi(M)} \alpha} = 0.$$

*Then $\left| N, \partial N \right|^{\alpha} = 0$. In particular, $\left| N, \partial N \right| = 0$.*

*Proof.* As in the proof of Proposition 4.4, one can take a small parametrised relative fundamental cycle of $(M, \partial M)$, and then use the uniform boundary condition on $T_1 \cong T_2$ to construct a small parametrised relative fundamental cycle of $(N, \partial N)$. □

## 4.3. **Proof of Theorem 1.6.**

*Proof of Theorem 1.6.* If $\Gamma$ is a graph of manifolds, then instead of performing all glueings in the geometric realisation $M(\Gamma)$ at once, we can also do them step by step, glueing one pair of boundary tori after another. Therefore, we can prove Theorem 1.6 by induction over the number of edges of $\Gamma$.

The base case of this induction is a graph of manifolds without edges, i.e., a disjoint union of vertex manifolds; this case is handled in Proposition 4.3.

In the induction step, we have to distinguish two cases:

- In case of a glueing corresponding to an edge connecting two different connected components of the underlying graph, we apply Proposition 4.4.
- In case of a glueing corresponding to an edge connecting vertices in the same connected component of the underlying graph (this includes the case of loops), we apply Proposition 4.5. □

### 4.4. **Proof of Theorem 1.2.**

*Proof of Theorem 1.2.* Let $\alpha$ be the standard $\pi_1(M)$-space given by the profinite completion of the residually finite group $\pi_1(M)$ [9, Section 2.1]. Then [19, Theorem 6.6, Remark 6.7]

$$\|M\|_{\mathbb{Z}}^{\infty} = \left| M \right|^{\alpha}.$$

On the other hand, $\left| M \right|^{\alpha} = 0$, by Theorem 1.6 (if $\pi_1(M)$ is residually finite, the action on the profinite completion is free). Therefore, $\|M\|_{\mathbb{Z}}^{\infty} = 0$. Because of the sandwich [19, Proposition 6.1]

$$0 \leq \|M\| \leq \|M\|_{\mathbb{Z}}^{\infty} = 0,$$

we also obtain $\|M\| = 0$.

In addition, if $\bigcap_{j \in \mathbb{N}} \Gamma_j = \{1\}$, then the action $\beta$ of $\pi_1(M)$ on the corresponding coset tree is a free standard $\pi_1(M)$-space and [9, Theorem 2.6]

$$\lim_{j \to \infty} \frac{\|M_j\|_{\mathbb{Z}}}{[\pi_1(M) : \Gamma_j]} = \left| M \right|^{\beta}.$$

Moreover, $\left| M \right|^{\beta} = 0$ by Theorem 1.6. □

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